647510 MSC1MatC102x Seat No :__________

M.Sc.(Maths.) Semester - 1 (CBCS) Examination

Jan/Feb.-2022 [NEW COURSE]

Real Analysis(Core (New))

Time: 1:30 Hours Marks: 42

Instructions:

1. Figures to the right indicate marks
2. There are five questions in the question paper
3. Answer any three of the following questions.

Q.1 (a) **Attempt any TWO.** **10**

(i) Define the outer measure $m^*(A)$ of $A \subset \mathbb{R}$. If A is countable, show that $m^*(A) = 0$.

(ii) If f is measurable and $f = g$ almost everywhere, prove that g is measurable.

(iii) Show that the function $f : [0, 1] \to \mathbb{R}$, defined by $f(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2} \\ \frac{1}{2} & \text{if } x = \frac{1}{2} \\ 1 & \text{otherwise} \end{cases}$ is measurable.

(b) **Attempt any ONE.** **4**

(i) Define σ-algebra of sets with a simple illustration.

(ii) Define Borel sets. Show that every F_σ set is a Borel set.

Q.2 (a) **Attempt any TWO.** 10

(i) Prove that the function defined on $[0, 1]$ by $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational, is Lebesgue integrable function on [0,1].

(ii) State and prove Fatou's Lemma.

(iii) Let f be a bounded measurable function defined on a set E with finite measure and if $A \leq f(x) \leq B$ for all $x \in E$, prove that

$A\, m(E) \leq \int_E f \leq B\, m(E)$.

(b) **Attempt any ONE.** 4

(i) Define convergence in measure. If (f_n) is a sequence of measurable functions defined on a set E of finite measure and if $f_n \to f$ almost everywhere on E, prove that $f_n \to f$ in measure.

(ii) For each $n \in \mathbb{N}$, define $f_n(x) = x^n$ on $[0, 1]$. Show that the sequence $f_n \to 0$ almost everywhere on [0,1].

Q.3 (a) **Attempt any TWO.** 10

(i) If f is integrable on $[a,b]$ and $\int_a^x f(t)dt = 0$, for all $x \in [a,b]$ then prove that $f(t) = 0$ almost everywhere on $[a,b]$.

(ii) Let $f(x) = x\sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$. Find $D^+f(0)$ and $D^-f(0)$.

(iii) If f is integrable on $[a,b]$, show that its indefinite integral is continuous on $[a,b]$.

(b) **Attempt any ONE.** 4

(i) Define absolutely continuous function on $[a,b]$. Prove that every absolutely continuous function on $[a,b]$ is uniformly continuous.

(ii) Is it true that $f(x) = \sin x$ is absolutely continuous on $[0, \frac{\pi}{2}]$? Justify.

Q.4 (a) **Attempt any TWO.** 10

(i) Give example of funcions f, g defined on $[0,1]$ such that $f \in L^1[0,1]$ but $g \notin L^1[0,1]$.

(ii) State and prove Minkowski's inequality in the space $L^p[0,1]$ $(1 \leq p < \infty)$.

(iii) Show that $L^p[0,1]$ $(1 \leq p < \infty)$ is a vector space.

(b) **Attempt any ONE.** 4

(i) If $f \in L^p[0,1]$ with $\|f\|_p = 0$, then $f(x) = 0$ for each $x \in [0,1]$. True or false? Justify.

(ii) For each $n \in \mathbb{N}$, define $f_n(x) = x^n$ on $[0,1]$. Show that the sequence f_n converges pointwise to a discontinuous function.

Q.5 (a) **Attempt any TWO.** 10

(i) Show that the set $A = \{1,2,3,4,5\} \subset \mathbb{R}$ is measurable. Find $m(A)$.

(ii) Show that Cantor set has measure zero.

(iii) Give an example of a continuous function on $[0,1]$ that is not absolutely continuous.

(b) **Attempt any ONE.** 4

(i) Find the the outer measure of $(0,1) \cup [2,3)$.

(ii) Prove that every Riemann integrable function f on $[0,1]$ is Lebesgue integrable.

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